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Research Article

An Application of Hybrid Steepest Descent Methods for Equilibrium Problems and Strict Pseudocontractions in Hilbert Spaces

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We use the hybrid steepest descent methods for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a strict pseudocontraction mapping in the setting of real Hilbert spaces. We proved strong convergence theorems of the sequence generated by our proposed schemes.

1. Introduction

Let H be a real Hilbert space and C a closed convex subset of H, and let ϕ be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $\phi : C \times C \to R$ is to find $x \in C$ such that

$$\mathrm{EP} : \phi(x, y) \geq 0 \quad \forall y \in C \tag{1.1}$$

denoted the set of solution by $\mathrm{EP}(\phi)$. Given a mapping $T : C \to H$, let $\phi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$, then $z \in \mathrm{EP}(\phi)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimizations, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem, see, for instance, [1, 2].

A mapping T of C into itself is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. In 2007, Plubtieng and Punpaeng [3], S. Takahashi and W. Takahashi [4], and Tada and W. Takahashi [5] considered iterative methods for finding an element of $\mathrm{EP}(\phi) \cap F(T)$.

Recall that an operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.2}$$

In 2006, Marino and Xu [6] introduced the general iterative method and proved that for a given $x_0 \in H$, the sequence $\{x_n\}$ is generated by the algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.3}$$

where T is a self-nonexpansive mapping on H, f is a contraction of H into itself with $\beta \in (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, and A is a strongly positive bounded linear operator on H and converges strongly to a fixed-point x^* of T which is the unique solution to the following variational inequality:

$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0$, for $x \in F(T)$, and is also the optimality condition for some minimization problem. A mapping $S : C \to H$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

Note that the class of k-strict pseudo-contraction strictly includes the class of nonexpansive mapping, that is, S is nonexpansive if and only if S is 0-srictly pseudocontractive; it is also said to be pseudocontractive if $k = 1$. Clearly, the class of k-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

The set of fixed points of S is denoted by $F(S)$. Very recently, by using the general approximation method, Qin et al. [7] obtained a strong convergence theorem for finding an element of $F(S)$. On the other hand, Ceng et al. [8] proposed an iterative scheme for finding an element of $\mathrm{EP}(\phi) \cap F(S)$ and then obtained some weak and strong convergence theorems. Based on the above work, Y. Liu [9] introduced two iteration schemes by the general iterative method for finding an element of $\mathrm{EP}(\phi) \cap F(S)$.

In 2001, Yamada [10] introduced the following hybrid iterative method for solving the variational inequality:

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \quad n \geq 0, \tag{1.5}$$

where F is k-Lipschitzian and η-strongly monotone operator with $k > 0, \eta > 0, 0 < \mu < 2\eta/k^2$, then he proved that if $\{\lambda_n\}$ satisfyies appropriate conditions, the $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of variational inequality

$$\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad \forall x \in F_{ix}(T), \quad \tilde{x} \in F_{ix}(T). \tag{1.6}$$

Motivated and inspired by these facts, in this paper, we introduced two iteration methods by the hybrid iterative method for finding an element of $\mathrm{EP}(\phi) \cap F(S)$, where $S : C \to H$ is a k-strictly pseudocontractive non-self mapping, and then obtained two strong convergence theorems.

2. Preliminaries

Throughout this paper, we always assume that C is a nonempty closed convex subset of a Hilbert space H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

Such a $P_C x$ is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$, $u = p_c x, \Leftrightarrow \langle x - u, u - y \rangle \geq 0$, for all $y \in C$.

It is widely known that H satisfies Opial's condition [11], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \tag{2.2}$$

holds for every $y \in H$ with $y \neq x$. In order to solve the equilibrium problem for a bifunction $\phi : C \times C \to R$, let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0$, for all $x \in C$,

(A2) ϕ is monotone, that is, $\phi(x, y) + \phi(y, x) \leq 0$, for all $x, y \in C$,

(A3) For all $x, y, z \in C$.

$$\lim_{t \downarrow 0} \phi(tz + (1 - t)x, y) \leq \phi(x, y); \tag{2.3}$$

(A4) For each fixed $x \in C$, the function $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 2.1 (see [12]). *Let ϕ be a bifunction from $C \times C$ into R satisfying (A1), (A2),(A3) and (A4) then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Further, if $T_r x = \{z \in C : \phi(z, y) + 1/r\langle y - z, z - x \rangle \geq 0, \forall y \in C\}$, then the following hold:

(1) *T_r is single-valued,*

(2) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H; \tag{2.5}$$

(3) *$F(T_r) = EP(\phi)$,*

(4) EP(ϕ) *is nonempty, closed and convex.*

Lemma 2.2 (see [13]). *If $S : C \to H$ is a k-strict pseudo-contraction, then the fixed-point set $F(S)$ is closed convex, so that the projection $P_{F(S)}$ is well difened.*

Lemma 2.3 (see [14]). *Let $S : C \to H$ be a k-strict pseudo-contraction. Define $T : C \to H$ by $Tx = \lambda x + (1 - \lambda)Sx$ for each $x \in C$, then, as $\lambda \in [k, 1)$, T is nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.4 (see [15]). *In a Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, (x + y)\rangle, \quad \forall x, y \in H. \tag{2.6}$$

Lemma 2.5 (see [16]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0, \tag{2.7}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$, such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.
Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

Throughout the rest of this paper, we always assume that F is a L-lipschitzian continuous and η-strongly monotone operator with $L, \eta > 0$ and assume that $0 < \mu < 2\eta/L^2$. $\tau = \mu(\eta - \mu L^2/2)$. Let $\{T_{\lambda_n}\}$ be mappings defined as Lemma 2.1. Define a mapping $S_n : C \to H$ by $S_n x = \beta_n x + (1 - \beta_n)Sx$, for all $x \in C$, where $\beta_n \in [k, 1)$, then, by Lemma 2.3, S_n is nonexpansive. We consider the mapping G_n on H defined by

$$G_n x = (I - \alpha_n\mu F)S_n T_{\lambda_n} x, \quad x \in H, \quad n \in N, \tag{3.1}$$

where $\alpha_n \in (0, 1)$. By Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \|G_n x - G_n y\| &\le (1 - \alpha_n\tau)\|T_{\lambda_n}x - T_{\lambda_n}y\| \\ &\le (1 - \alpha_n\tau)\|x - y\|. \end{aligned} \tag{3.2}$$

It is easy to see that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed-point $x_n^F \in H$ such that

$$x_n^F = (I - \alpha_n\mu F)S_n T_{\lambda_n} x_n^F. \tag{3.3}$$

For simplicity, we will write x_n for x_n^F provided no confusion occurs. Next, we prove that the sequence $\{x_n\}$ converges strongly to a $q \in F(S) \cap \mathrm{EP}(\phi)$ which solves the variational inequality

$$\langle Fq, p - q \rangle \geq 0, \quad \forall p \in F(S) \cap \mathrm{EP}(\phi). \tag{3.4}$$

Equivalently, $q = P_{F(S)\cap \mathrm{EP}(\phi)}(I - \mu F)q$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3), and (A4). Let $S : C \to H$ be a k-strictly pseudocontractive nonself mapping such that $F(S) \cap \mathrm{EP}(\phi) \neq \phi$. Let $F : H \to H$ be an L-Lipschitzian continuous and η-strongly monotone operator on H with $L, \eta > 0$ and $0 < \mu < 2\eta/L^2$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{x_n\}$ be asequence generated by*

$$\begin{gathered} \phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n) S u_n, \\ x_n = (I - \alpha_n \mu F) y_n, \quad \forall n \in N, \end{gathered} \tag{3.5}$$

where $u_n = T_{\lambda_n} x_n$, $y_n = S_n u_n$, and $\{\lambda_n\} \subset (0, +\infty)$ satisfy $\liminf_{n\to\infty} \lambda_n > 0$ if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 \leq k \leq \beta_n \leq \lambda < 1$ *and* $\lim_{n\to\infty} \beta_n = \lambda$,

then $\{x_n\}$ converges strongly to a point $q \in F(S) \cap \mathrm{EP}(\phi)$ which solves the variational inequality (3.4).

Proof. First, take $p \in F(S) \cap \mathrm{EP}(\phi)$. Since $u_n = T_{\lambda_n} x_n$ and $p = T_{\lambda_n} p$, from Lemma 2.1, for any $n \in N$, we have

$$\|u_n - p\| = \|T_{\lambda_n} x_n - T_{\lambda_n} p\| \leq \|x_n - p\|. \tag{3.6}$$

Then, since $S_n p = p$, we obtain that

$$\|y_n - p\| = \|S_n u_n - S_n p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.7}$$

Further, we have

$$\begin{aligned} \|x_n - p\| &= \|-\alpha_n \mu F p + (I - \mu\alpha_n F) y_n - (I - \mu\alpha_n F) p\| \\ &\leq \alpha_n \|-\mu F(p)\| + (1 - \alpha_n \tau)\|y_n - p\|. \end{aligned} \tag{3.8}$$

It follows that $\|x_n - p\| \leq \|\mu F(p)\|/\tau$.

Hence, $\{x_n\}$ is bounded, and we also obtain that $\{u_n\}$ and $\{y_n\}$ are bounded. Notice that

$$\begin{aligned}\|u_n - y_n\| &\le \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|u_n - x_n\| + \alpha_n \|-\mu F y_n\|.\end{aligned} \tag{3.9}$$

By Lemma 2.1, we have

$$\begin{aligned}\|u_n - p\|^2 = \|T_{\lambda_n} x_n - T_{\lambda_n} p\|^2 &\le \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2\Big).\end{aligned} \tag{3.10}$$

It follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.11}$$

Thus, from Lemma 2.4, (3.7), and (3.11), we obtain that

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n(-\mu F p) + (I - \mu\alpha_n F) y_n - (I - \mu\alpha_n F) p\|^2 \\ &\le (1 - \alpha_n\tau)^2 \|y_n - p\|^2 + 2\alpha_n \langle -\mu F p, x_n - p\rangle \\ &\le (1 - \alpha_n\tau)^2 \|u_n - p\|^2 + 2\alpha_n \langle -\mu F p, x_n - p\rangle \\ &\le (1 - \alpha_n\tau)^2 \Big(\|x_n - p\|^2 - \|x_n - u_n\|^2\Big) + 2\alpha_n \|-\mu F p\| \|x_n - p\| \\ &= \Big(1 - 2\alpha_n\tau + (\alpha_n\tau)^2\Big)\|x_n - p\|^2 \\ &\quad - (1 - \alpha_n\tau)^2 \|x_n - u_n\|^2 + \|2\alpha_n\| - \mu F p\|x_n - p\| \\ &\le \|x_n - p\|^2 + (\alpha_n\tau)^2 \|x_n - p\|^2 - (1 - \alpha_n\tau)^2 \|x_n - u_n\|^2 + 2\alpha_n \|-\mu F p\| \|x_n - p\|.\end{aligned} \tag{3.12}$$

It follows that

$$(1 - \alpha_n\tau)^2 \|x_n - u_n\|^2 \le (\alpha_n\tau)^2 \|x_n - p\|^2 + 2\alpha_n \|\mu F p\| \|x_n - p\|. \tag{3.13}$$

Since $\alpha_n \to 0$, therefore

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.14}$$

From (3.9), we derive that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.15}$$

Define $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$, then T is nonexpansive with $F(T) = F(S)$ by Lemma 2.3. We note that

$$\|Tu_n - u_n\| \leq \|Tu_n - y_n\| + \|y_n - u_n\| \leq |\lambda - \beta_n| \|u_n - Su_n\| + \|y_n - u_n\|. \tag{3.16}$$

So by (3.15) and $\beta_n \to \lambda$, we obtain that

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0. \tag{3.17}$$

Since $\{u_n\}$ is bounded, so there exists a subsequence $\{u_{n_i}\}$ which converges weakly to q. Next, we show that $q \in F(S) \cap \mathrm{EP}(\phi)$. Since C is closed and convex, C is weakly closed. So we have $q \in C$. Let us show that $q \in F(S)$. Assume that $q \overline{\in} F(T)$, Since $u_{n_i} \rightharpoonup q$ and $q \neq Tq$, it follows from the Opial's condition that

$$\begin{aligned}
\liminf_{n\to\infty} \|u_{n_i} - q\| &< \liminf_{n\to\infty} \|u_{n_i} - Tq\| \\
&\leq \liminf_{n\to\infty} \left(\|u_{n_i} - Tu_{n_i}\| + \|Tu_{n_i} - Tq\| \right) \\
&\leq \liminf_{n\to\infty} \|u_{n_i} - q\|.
\end{aligned} \tag{3.18}$$

This is a contradiction. So, we get $q \in F(T)$ and $q \in F(S)$.

Next, we show that $q \in EP(\phi)$. Since $u_n = T_{\lambda_n} x_n$, for any $y \in C$, we obtain

$$\phi(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0. \tag{3.19}$$

From (A2), we have

$$\frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq \phi(y, u_n). \tag{3.20}$$

Replacing n by n_i, we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{\mathrm{n}_i}}{\lambda_{n_i}} \right\rangle \geq \phi(y, u_{n_i}). \tag{3.21}$$

Since $(u_{n_i} - x_{n_i})/\lambda_{n_i} \to 0$ and $u_{n_i} \rightharpoonup q$, it follows from (A4) that $0 \geq \phi(y, q)$, for all $y \in C$. Let $z_t = ty + (1-t)q$ for all $t \in (0,1]$ and $y \in C$, then we have $z_t \in C$ and hence $\phi(z_t, q) \leq 0$. Thus, from (A1) and (A4), we have

$$0 = \phi(z_t, z_t) \leq t\phi(z_t, y) + (1-t)\phi(z_t, q) \leq t\phi(z_t, y), \tag{3.22}$$

and hence $0 \leq \phi(z_t, y)$. From (A3), we have $0 \leq \phi(q, y)$ for all $y \in C$ and hence $q \in \mathrm{EP}(\phi)$. Therefore, $q \in F(S) \cap \mathrm{EP}(\phi)$. On the other hand, we note that

$$x_n - q = -\alpha_n \mu F q + (I - \mu\alpha_n F) y_n - (I - \mu\alpha_n F) q. \tag{3.23}$$

Hence, we obtain

$$\begin{aligned} \|x_n - q\|^2 &= \langle -\alpha_n \mu F q, x_n - q\rangle + \langle (I - \mu\alpha_n F) y_n - (I - \mu\alpha_n F) q, x_n - q\rangle \\ &\leq \alpha_n \langle -\mu F q, x_n - q\rangle + (1 - \alpha_n \tau)\|x_n - q\|^2. \end{aligned} \tag{3.24}$$

It follows that

$$\|x_n - q\|^2 \leq \frac{1}{\tau}\langle -\mu F q, x_n - q\rangle. \tag{3.25}$$

This implies that

$$\|x_n - q\|^2 \leq \frac{\langle -\mu F q, x_n - q\rangle}{\tau}. \tag{3.26}$$

In particular,

$$\|x_{n_i} - q\|^2 \leq \frac{\langle -\mu F q, x_{n_i} - q\rangle}{\tau}. \tag{3.27}$$

Since $x_{n_i} \rightharpoonup q$, it follows from (3.27) that $x_{n_i} \to q$ as $i \to \infty$. Next, we show that q solves the variational inequality (3.4).

As a matter of fact, we have

$$\begin{aligned} x_n &= (I - \alpha_n \mu F) y_n \\ &= (I - \alpha_n \mu F) S_n T_{\lambda_n} x_n, \end{aligned} \tag{3.28}$$

and we have

$$\mu F x_n = -\frac{1}{\alpha_n}\left\{(I - S_n T_{\lambda_n}) x_n - \mu\alpha_n (F x_n - F S_n T_{\lambda_n} x_n)\right\}. \tag{3.29}$$

Hence, for $p \in F(S) \cap \mathrm{EP}(\phi)$,

$$\begin{aligned} \langle (\mu F) x_n, x_n - p\rangle &= -\frac{1}{\alpha_n}\left\langle \left\{(I - S_n T_{\lambda_n}) x_n - \mu\alpha_n (F x_n - F S_n T_{\lambda_n} x_n)\right\}, x_n - p\right\rangle \\ &= -\frac{1}{\alpha_n}\langle (I - S_n T_{\lambda_n}) x_n - (I - S_n T_{\lambda_n}) p, x_n - p\rangle + \mu\langle (F x_n - F S_n T_{\lambda_n} x_n), x_n - p\rangle. \end{aligned} \tag{3.30}$$

Since $I - S_n T_{\lambda_n}$ is monotone (i.e., $\langle x-y, (I-S_nT_{\lambda_n})x - (I-S_nT_{\lambda_n})y\rangle \geq 0$, for all $x, y \in H$. This is due to the nonexpansivity of $S_nT_{\lambda_n}$).

Now replacing n in (3.30) with n_i and letting $i \to \infty$, we obtain

$$\begin{aligned}\langle (\mu F)q, q-p\rangle &= \lim_{i\to\infty}\langle \mu F x_{n_i}, x_{n_i} - p\rangle \\ &\leq \lim_{i\to\infty}\mu\langle F x_{n_i} - F S_n T_{\lambda_n} x_{n_i}, x_{n_i} - p\rangle = 0.\end{aligned} \tag{3.31}$$

That is, $q \in F(S)\cap \mathrm{EP}(\phi)$ is a solution of (3.4). To show that the sequence $\{x_n\}$ converges strongly to q, we assume that $x_{n_k} \to \widehat{x}$. Similiary to the proof above, we derive $\widehat{x} \in F(S) \cap \mathrm{EP}(\phi)$. Moreover, it follows from the inequality (3.31) that

$$\langle (\mu F)q, q-\widehat{x}\rangle \leq 0. \tag{3.32}$$

Interchange q and $\widehat{x}$ to obtain

$$\langle (\mu F)\widehat{x}, \widehat{x}-q\rangle \leq 0. \tag{3.33}$$

Adding up (3.32) and (3.33) yields

$$(\mu\eta)\|q-\widehat{x}\|^2 \leq \langle q-\widehat{x}, (\mu F)q - (\mu F)\widehat{x}\rangle \leq 0. \tag{3.34}$$

Hence, $q = \widehat{x}$, and therefore $x_n \to q$ as $n \to \infty$,

$$\langle (I-\mu F)q - q, q-p\rangle \geq 0, \forall p \in F(S)\cap \mathrm{EP}(\phi). \tag{3.35}$$

This is equivalent to the fixed-point equation

$$P_{F(S)\cap \mathrm{EP}(\phi)}(I-\mu F)q = q. \tag{3.36}$$

□

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3) and (A4). Let $S : C \to H$ be a k-strictly pseudocontractive nonself mapping such that $F(S)\cap \mathrm{EP}(\phi) \neq \phi$. Let $F : H \to H$ be an L-Lipschitzian continuous and η-strongly monotone operator on H with $L, \eta > 0$. Suppose that $0 < \mu < 2\eta/L^2$, $\tau = \mu(\eta - \mu L^2/2)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{gathered}\phi(u_n, y) + \frac{1}{\lambda_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n u_n + (1-\beta_n)S u_n, \\ x_{n+1} = (I - \alpha_n \mu F)y_n, \quad \forall n \in N,\end{gathered} \tag{3.37}$$

where $u_n = T_{\lambda_n}x_n$, $y_n = S_n u_n$ *if* $\{\alpha_n\}$,$\{\beta_n\}$, *and* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$,

(ii) $0 \le k \le \beta_n \le \lambda < 1$ *and* $\lim_{n\to\infty}\beta_n = \lambda$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(iii) $\{\lambda_n\} \in (0,+\infty)$, $\lim_{n\to\infty}\lambda_n > 0$ *and* $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $q \in F(S)\cap \mathrm{EP}(\phi)$ *which solves the variational inequality*(3.4).

Proof. We first show that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(S)\cap \mathrm{EP}(\phi)$ to derive that

$$\begin{aligned}\|x_{n+1} - p\| &= \|-\alpha_n\mu Fp + (I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)p\| \\ &\le \alpha_n\|-\mu F(p)\| + (1 - \alpha_n\tau)\|x_n - p\| \\ &\le (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\|-\mu F(p)\|.\end{aligned} \tag{3.38}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{1}{\tau}\| - \mu F(p)\|\right\}, \quad \forall n \in N, \tag{3.39}$$

and hence $\{x_n\}$ is bounded. From (3.6) and (3.7), we also derive that $\{u_n\}$ and $\{y_n\}$ are bounded. Next, we show that $\|x_{n+1} - x_n\| \to 0$. We have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(I - \alpha_n\mu F)y_n - (I - \alpha_{n-1}\mu F)y_{n-1}\| \\ &= \|(I - \alpha_n\mu F)y_n - (I - \alpha_n\mu F)y_{n-1} + (I - \alpha_n\mu F)y_{n-1} - (I - \alpha_{n-1}\mu F)y_{n-1}\| \\ &\le (1 - \alpha_n\tau)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\mu F y_{n-1}\| \\ &\le (1 - \alpha_n\tau)\|y_n - y_{n-1}\| + K|\alpha_n - \alpha_{n-1}|,\end{aligned} \tag{3.40}$$

where

$$K = \sup\{\|\mu F y_n\| : n \in N\} < \infty. \tag{3.41}$$

On the other hand, we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|S_n u_n - S_{n-1}u_{n-1}\| \\ &\le \|S_n u_n - S_n u_{n-1}\| + \|S_n u_{n-1} - S_{n-1}u_{n-1}\| \\ &\le \|u_n - u_{n-1}\| + \|S_n u_{n-1} - S_{n-1}u_{n-1}\|.\end{aligned} \tag{3.42}$$

From $u_{n+1} = T_{\lambda_{n+1}} x_{n+1}$ and $u_n = T_{\lambda_n} x_n$, we note that

$$\phi(u_{n+1}, y) + \frac{1}{\lambda_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C, \tag{3.43}$$

$$\phi(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.44}$$

Putting $y = u_n$ in (3.43) and $y = u_{n+1}$ in (3.44), we have

$$\begin{gathered} \phi(u_{n+1}, u_n) + \frac{1}{\lambda_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \\ \phi(u_n, u_{n+1}) + \frac{1}{\lambda_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0. \end{gathered} \tag{3.45}$$

So, from (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n+1} - x_{n+1}}{\lambda_{n+1}} \right\rangle \geq 0\ , \tag{3.46}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{\lambda_n}{\lambda_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.47}$$

Since $\lim_{n\to\infty} \lambda_n > 0$, without loss of generality, let us assume that there exists a real number a such that $\lambda_n > a > 0$ for all $n \in N$. Thus, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\} \\ \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{a} |\lambda_{n+1} - \lambda_n| M_0, \end{aligned} \tag{3.48}$$

where $M_0 = \sup\{\|u_n - x_n\| : n \in N\}$. Next, we estimate $\|S_n u_{n-1} - S_{n-1} u_{n-1}\|$. Notice that

$$\begin{aligned} \|S_n u_{n-1} - S_{n-1} u_{n-1}\| &= \left\| (\beta_n u_{n-1} + (1 - \beta_n) S u_{n-1}) - (\beta_{n-1} u_{n-1} + (1 - \beta_{n-1}) S u_{n-1}) \right\| \\ &\leq \left|\beta_n - \beta_{n-1}\right| \|u_{n-1} - S u_{n-1}\|. \end{aligned} \tag{3.49}$$

From (3.48), (3.49), and (3.42), we obtain that

$$\begin{aligned} \left\|y_n - y_{n-1}\right\| &\leq \|x_n - x_{n-1}\| + \frac{M_0}{a} |\lambda_n - \lambda_{n-1}| + \left|\beta_n - \beta_{n-1}\right| \|u_{n-1} - S u_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| M_1 + \left|\beta_n - \beta_{n-1}\right| M_1, \end{aligned} \tag{3.50}$$

where M_1 is an appropriate constant such that

$$M_1 \geq \frac{M_0}{a} + \|u_{n-1} - Su_{n-1}\|, \quad \forall n \in N. \tag{3.51}$$

From (3.41) and (3.50), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq K|\alpha_n - \alpha_{n-1}| + (1 - \alpha_n\tau)\left(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 + \left|\beta_n - \beta_{n-1}\right|M_1\right) \\ &\leq (1 - \alpha_n\tau)\|x_n - x_{n-1}\| + M\left(|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}| + \left|\beta_n - \beta_{n-1}\right|\right),\end{aligned} \tag{3.52}$$

where $M = \max[K, M_1]$. Hence, few by Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.53}$$

From (3.48) and (3.50), $|\lambda_n - \lambda_{n-1}| \to 0$ and $|\beta_n - \beta_{n-1}| \to 0$, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0, \qquad \lim_{n\to\infty} \left\|y_{n+1} - y_n\right\| = 0. \tag{3.54}$$

Since

$$x_{n+1} = \left(I - \alpha_n\mu F\right)y_n, \tag{3.55}$$

it follows that

$$\begin{aligned}\left\|x_n - y_n\right\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\left\|-\mu F y_n\right\|.\end{aligned} \tag{3.56}$$

From $\alpha_n \to 0$ and (3.53), we have

$$\lim_{n\to\infty} \left\|x_n - y_n\right\| = 0. \tag{3.57}$$

For $p \in F(S) \cap \mathrm{EP}(\phi)$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \left\|T_{\lambda_n}x_n - T_{\lambda_n}p\right\|^2 \leq \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}\left(\left\|x_n - p\right\|^2 + \left\|u_n - p\right\|^2 - \|u_n - x_n\|^2\right).\end{aligned} \tag{3.58}$$

This implies that

$$\left\|u_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \|u_n - x_n\|^2. \tag{3.59}$$

Then, from (3.7) and (3.59), we derive that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|-\mu\alpha_n Fp + (I-\mu\alpha_n F)y_n - (I-\mu\alpha_n F)p\|^2 \\ &\le (1-\alpha_n\tau)^2\|y_n-p\|^2 + \alpha_n^2\|-\mu Fp\|^2 + 2\alpha_n\|-\mu Fp\|\|y_n-p\| \\ &\le \|u_n-p\|^2 + \alpha_n^2\|-\mu Fp\|^2 + 2\alpha_n\|-\mu Fp\|\|y_n-p\| \\ &\le \|x_n-p\|^2 - \|x_n-u_n\|^2 + \alpha_n^2\|-\mu Fp\|^2 + 2\alpha_n\|-\mu Fp\|\|y_n-p\|.\end{aligned} \tag{3.60}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, we have

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.61}$$

From (3.57) and (3.61), we obtain that

$$\|u_n-y_n\| \le \|u_n-x_n\| + \|x_n-y_n\| \to 0, \quad \text{as } n\to\infty. \tag{3.62}$$

Define $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$, then T is nonexpansive with $F(T) = F(S)$ by Lemma 2.3. Notice that

$$\begin{aligned}\|Tu_n-u_n\| &\le \|Tu_n-y_n\| + \|y_n-u_n\| \\ &\le |\lambda-\beta_n|\|u_n-Su_n\| + \|y_n-u_n\|.\end{aligned} \tag{3.63}$$

By (3.62) and $\beta_n \to \lambda$, we obtain that

$$\lim_{n\to\infty}\|Tu_n-u_n\| = 0. \tag{3.64}$$

Next, we show that $\limsup_{n\to\infty}\langle \mu Fq, q-x_n\rangle \le 0$, where $q = P_{F(S)\cap \mathrm{EP}(\phi)}(I-\mu F)q$ is a unique solution of the variational inequality (3.4). Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle \mu Fq, q-x_{n_i}\rangle = \limsup_{n\to\infty}\langle \mu Fq, q-x_n\rangle. \tag{3.65}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w.

Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From (3.61) and (3.64), we obtain $x_{n_i} \rightharpoonup w$ and $Tu_{n_i} \rightharpoonup w$. By the same argument as in the proof of Theorem 3.1, we have $w \in F(S)\cap \mathrm{EP}(\phi)$. Since $q = P_{F(S)\cap\mathrm{EP}(\phi)}(I-\mu F)q$, it follows that

$$\limsup_{n\to\infty}\langle \mu Fq, q-x_n\rangle = \langle \mu Fq, q-w\rangle \le 0. \tag{3.66}$$

From $x_{n+1} - q = -\alpha_n \mu F q + (I - \mu\alpha_n F) y_n - (I - \mu\alpha_n F) q$, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\le \|(I - \mu\alpha_n F) y_n - (I - \mu\alpha_n F) q\|^2 + 2\alpha_n \langle -\mu F q, x_{n+1} - q\rangle \\ &\le (1 - \alpha_n \tau)^2 \|x_n - q\|^2 + 2\alpha_n \langle -\mu F q, x_{n+1} - q\rangle. \end{aligned} \tag{3.67}$$

This implies that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\le \left\{1 - 2\alpha_n\tau + (\alpha_n\tau)^2\right\} \|x_n - q\|^2 + 2\alpha_n \langle -\mu F q, x_{n+1} - q\rangle \\ &= (1 - 2\alpha_n\tau)\|x_n - q\|^2 + (\alpha_n\tau)^2 \|x_n - q\|^2 + 2\alpha_n \langle -\mu F q, x_{n+1} - q\rangle \\ &= (1 - 2\alpha_n\tau)\|x_n - q\|^2 + 2\alpha_n\tau \left\{ \frac{\alpha_n\tau^2}{2\tau} M^* + \frac{1}{\tau}\langle -\mu F q, x_{n+1} - q\rangle \right\} \\ &= (1 - \gamma_n)\|x_n - q\|^2 + \gamma_n \delta_n, \end{aligned} \tag{3.68}$$

where $M^* = \sup\{\|x_n - q\|^2 : n \in N\}$, $\gamma_n = 2\alpha_n\tau$, and $\delta_n = (\alpha_n\tau^2/2\tau)M^* + (1/\tau)\langle -\mu F q, x_{n+1} - q\rangle$.

It is easy to see that $\gamma_n \to 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and $\limsup_{n\to\infty} \delta_n \le 0$ by (3.66). Hence by Lemma 2.5, the sequence $\{x_n\}$ converges strongly to q. □

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